Semigroups with the ascending chain condition on (principal) right ideals

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Ascending chain condition

A *finiteness condition* for a class of universal algebras is a property that is satisfied by at least all finite members of that class.

We call a semigroup S *weakly right noetherian* if it satisfies the ascending chain condition on right ideals; that is, every ascending chain

$$I_1 \subseteq I_2 \subseteq \cdots$$

of right ideals eventually terminates.

We shall consider the finiteness conditions of being weakly right noetherian and satifying the ascending chain condition on *principal* right ideals (ACCPR).

Equivalent formulations

Proposition. The following are equivalent for a semigroup S:

1. S is weakly right noetherian;
2. every non-empty set of right ideals of S has a maximal element;
3. ever right ideal of S is finitely generated.

Proposition. A monoid M is weakly right noetherian if and only if every finitely generated right M-act A is noetherian (i.e. A satisfies ACC on subacts).

Proposition. The following are equivalent for a semigroup S:

1. S satisfies ACCPR;
2. every non-empty set of principal right ideals of S has a maximal element.

Equivalent formulations

Theorem. The following are equivalent for a semigroup S:

1. S is weakly right noetherian;
2. S satisfies ACCPR and contains no infinite antichain of principal right ideals (under $\subseteq$);
3. S contains no infinite ascending chain or infinite antichain of $\mathcal{R}$-classses.

Sketch of proof. (1) $\Rightarrow$ (2). Clearly S satisfies ACCPR. If there exists an antichain $\{a_i S^1 : i \in \mathbb{N}\}$, then $a_1 S^1 \subsetneq \{a_1, a_2\}S^1 \subsetneq \{a_1, a_2, a_3\}S^1 \subsetneq \cdots$.
(2) $\Rightarrow$ (1). Suppose S is not w. r. n. but satifies ACCPR. There exists $I_1 \subsetneq I_2 \subsetneq \cdots$. Choose $a_1 \in I_1$ and $a_k \in I_k \setminus I_{k-1}$ for $k \geq 2$. Then $a_k S^1 \notin a_j S^1$ for $j < k$.
The set $\{a_i S^1 : i \in \mathbb{N}\}$ contains a maximal element, say $a_{k_1} S^1$; that is, $a_{k_1} S^1 \notin a_j S^1, j \neq k_1$.
The set $\{a_i S^1 : i \geq k_1 + 1\}$ contains a maximal element, say $a_{k_2} S^1$. Then $a_{k_2} S^1 \notin a_j S^1, j \in \mathbb{N} \setminus \{k_2\}$. Continuing in this way, we obtain an infinite antichain $\{a_{k_i} S^1 : i \in \mathbb{N}\}$.

Examples

Any semigroup with finitely many $\mathcal{R}$-classes is weakly right noetherian.

In the bicyclic monoid every right (and left ideal) is principal, so it is weakly right (and left) noetherian.

Every finitely generated commutative semigroup is *noetherian* (satisfies ACC on congruences) and hence weakly noetherian.

Every free semigroup F_X satisfies ACCPR, since $u <_{\mathcal{R}} v \Leftrightarrow v$ is a proper prefix of u.
F_X is weakly right noetherian iff $|X| = 1$. Indeed, if x, y are distinct element of X, then $\{\{x^i y\} : i \in \mathbb{N}\}$ is an infinite antichain of $\mathcal{R}$-classes.

Polycyclic monoid: $P_X = \langle X, X^{-1} \,|\, xx^{-1} = 1, xy^{-1} = 0 \, (x, y \in X, x \neq y) \rangle$.
P_X satisfies ACCPR for any X, but P_X is w. r. n. iff $|X| = 1$.

Quotients

Lemma. Let S be a semigroup and let T be a homomorphic image of S. If S is weakly right noetherian, then so is S.

Remark. The property of satisfying ACCPR is not closed under homomorphic images. Indeed, any free semigroup satisfies ACCPR.

Proposition. Let S be a semigroup and let $\rho \subseteq \mathcal{R}$ be a congruence on S. Then S is weakly right noetherian (resp. satifies ACCPR) if and only if S/ρ is weakly right noetherian (resp. satisfies ACCPR).

Ideals

Proposition. Let S be a semigroup and let I be an ideal of S. If both I and the Rees quotient S/I are weakly right noetherian, then so is S.

Proposition. Let S be a semigroup and let I be an ideal of S. If S satifies ACCPR, then so do both I and S/I.

Proof for I. $a_1 I^1 \subseteq a_2 I^1 \subseteq \cdots \Rightarrow a_1 S^1 \subseteq a_2 S^1 \subseteq \cdots$. There exists $n \in \mathbb{N}$ such that $a_m S^1 = a_n S^1$, $m \geq n$, so $a_m = a_n s_m, s_m \in S^1$. There exist $u_m \in I^1$ such that $a_n = a_m u_m$. Then $a_m = a_n(s_m u_m s_m) \in a_n I^1$. Thus $a_m I^1 = a_n I^1$.

Remark. The property is being weakly right noetherian is not closed under ideals.

Open problem. If S is a weakly right noetherian semigroup with a minimal ideal K, is K weakly right noetherian?

Construction

Let S be a semigroup, let $\theta : S \to T$ be a surjective homomorphism, and let $N = \{x_t : t \in T\} \cup \{0\}$ be a null semigroup. Define a multiplication on $S \cup N$, extending those on S and N, as follows:

$$sx_t = x_{(s\theta)t},\ x_t s = x_{t(s\theta)},\ s0 = 0s = 0.$$

With this multiplication $S \cup N$ is a semigroup, denoted by $\mathcal{U}(S, T, \theta)$.

Proposition. Let $U = \mathcal{U}(S, T, \theta)$.

1. U is weakly right noetherian iff S is weakly right noetherian.
2. U satifies ACCPR iff both S and T satify ACCPR.

Remark. Infinite null semigroups satify ACCPR but are not weakly right noetherian. Let T be a semigroup that does *not* satify *ACCPR*, and let $\theta : S \to T$ be a surj. hom. where S satisfies ACCPR (e.g. a free semigroup). Then U does not satify ACCPR, but both N and $U \setminus N = S$ do.

Subsemigroups

Lemma. Let S be a semigroup that is a union of subsemigroups $S_1, \ldots, S_n$. If all the S_i are weakly right noetherian, then so is S.

Example. Let $S = \langle a, b, c \mid abc = b \rangle_{Sgp}$. Then $S \leq \langle a, b, c \mid abcb^{-1} = 1 \rangle_{Gp}$. However, S does not satify ACCPR: $b <_{\mathcal{R}} ab <_{\mathcal{R}} a^2 b <_{\mathcal{R}} \cdots$.

We say that a subsemigroup T of a semigroup S is *$\mathcal{R}$-preserving* (in S) if for all $a, b \in T$ we have $aS^1 \subseteq bS^1 \Rightarrow aT^1 \subseteq bT^1$.
E.g. If T is regular, or if $S \setminus T$ is a left ideal.

Proposition. Let S be a semigroup and let T be an $\mathcal{R}$-preserving subsemigroup of S. If S is weakly right noetherian (resp. satifies ACCPR), then T is weakly right noetherian (resp. satisfies ACCPR).

Corollary. S is w. r. n. (resp. satisfies ACCPR) iff S^1 is w. r. n. (resp. satisfies ACCPR) iff S^0 is w. r. n. (resp. satisfies ACCPR).

Direct products

Let S be a semigroup and let $a \in S$. We say that a has a *local right identity* if $a \in aS$, i.e. there exists $b \in S$ such that $a = ab$.

Theorem. Let S and T be semigroups with S infinite.

1. Suppose T is infinite. Then $S \times T$ is w. r. n. iff both S and T are w. r. n. and every element of both S and T has a local right identity.
2. Suppose T is finite. Then $S \times T$ is w. r. n. iff S is w. r. n. and every element of T has a local right identity.

Theorem. Let S and T be semigroups. Then $S \times T$ satifies ACCPR iff one of the following holds:

1. both S and T satify ACCPR;
2. S satisfies ACCPR and has *no* element with a local right identity;
3. T satisfies ACCPR and has *no* element with a local right identity.

Direct products

Theorem. Let S and T be semigroups. Then $S \times T$ satifies ACCPR iff one of the following holds:

1. both S and T satify ACCPR;
2. S satisfies ACCPR and has *no* element with a local right identity;
3. T satisfies ACCPR and has *no* element with a local right identity.

Corollary. Let F be free semigroup. Then $F \times S$ satisfies ACCPR for any semigroup S.

Corollary. Let S be a semigroup and let T be a finite semigroup. Then $S \times T$ satisfies ACCPR iff S is satisfies ACCPR.

Free products

Let S and T be semigroups defined by presentations $\langle X \mid Q \rangle$ and $\langle Y \mid R \rangle$, respectively. The *semigroup free product* of S and T, denoted by $S * T$, is the semigroup defined by the presentation $\langle X, Y \mid Q, R \rangle$.
If S and T are monoids, then the *monoid free product* of S and T, denoted by $S *_1 T$, is defined by the presentation $\langle X, Y \mid Q, R, 1_S = 1_T \rangle$.

Theorem. Let S and T be semigroups. Then $S * T$ is weakly right noetherian iff both S and T are trivial.

Theorem. Let S and T be monoids. Then $S *_1 T$ is weakly right noetherian iff one of the following holds:

1. S is weakly right noetherian and T is trivial, or vice versa;
2. both S and T contain precisely two elements;
3. both S and T are groups.

Theorem. Let S and T be semigroups (resp. monoids). Then $S * T$ (resp. $S *_1 T$) satisfies ACCPR iff both S and T satisfy ACCPR.

Semilattices of semigroups

Let Y be a semilattice and let $(S_\alpha)_{\alpha \in Y}$ be a family of disjoint semigroups, indexed by Y.
If $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semigroup such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$, then S is called a *semilattice of semigroups*, and we denote it by $S = \mathcal{S}(Y, S_\alpha)$.

Now let $S = \bigcup_{\alpha \in Y} S_\alpha$, and suppose that for each $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there exists a homomorphism $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$. Furthermore, assume that:

- for each $\alpha \in Y$, the homomorphism $\phi_{\alpha,\alpha}$ is the identity map on S_α;
- for each $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, we have $\phi_{\alpha,\beta}\,\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.

For $a \in S_\alpha$ and $b \in S_\beta$, we define

$$ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}).$$

With this multiplication, S is a semilattice of semigroups. In this case we call S a *strong semilattice of semigroups* and denote it by $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$.

Semilattices of semigroups

Proposition. Let Y be a semilattice. Then Y is weakly noetherian iff it contains no infinite ascending chain or infinite antichain of elements.

Lemma. If $\mathcal{S}(Y, S_\alpha)$ is weakly right noetherian, then so is Y.

Example. Let Y be a semilattice and let F be a free semigroup. For each $\alpha \in Y$, let $F_\alpha = \{u_\alpha : u \in F\} \cong F$, and define $\phi_{\alpha,\beta} : F_\alpha \to F_\beta, u_\alpha \mapsto u_\beta$. Then $\mathcal{S}(Y, F_\alpha, \phi_{\alpha,\beta}) \cong F \times Y$ satisfies ACCPR.

Remark. $\mathcal{S}(Y, S_\alpha)$ w. r. n. $\not\Longrightarrow$ all S_α w. r. n.

Lemma. If $\mathcal{S}(Y, S_\alpha)$ satisfies ACCPR, then so do all the S_α.

Proposition. If Y satisfies ACCP, then $\mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$ satisfies ACCPR iff all the S_α satisfy ACCPR.

Regular semigroups

Proposition. A regular semigroup S is weakly right noetherian (resp. satisfies ACCPR) iff $\langle E(S) \rangle$ is weakly right noetherian (resp. satisfies ACCPR).

Corollary. An inverse semigroup S is weakly right noetherian (resp. satisfies ACCPR) iff $E(S)$ is weakly noetherian (resp. satisfies ACCP).

Corollary. An inverse semigroup is weakly right noetherian (resp. satisfies ACCPR) it is weakly left noetherian (resp. satisfies ACCPL).

Corollary. A Clifford semigroup $\mathcal{S}(Y, G_\alpha)$ is weakly right noetherian (resp. satisfies ACCPR) iff Y is weakly noetherian (resp. satisfies ACCP).

Completely regular semigroups

A *completely regular semigroup* is a union of groups. Every completely regular semigroup is a semilattice of completely simple semigroups.

Lemma. Let S be a completely simple semigroup.

1. S satisfies ACCPR.
2. S is weakly right noetherian iff it has finitely many $\mathcal{R}$-classes.

Example. Let $Y = (\mathbb{N}, \max)$, let $S_i = \{x_i, y_i\}$ be disjoint copies of the 2-element left zero semigroup, let $\phi_{i,i}$ be the identity map on S_i, and for $i < j$ define $\phi_{i,j} : S_i \to S_j, x_i, y_i \mapsto x_j$. Then $S = \mathcal{S}(Y, S_i, \phi_{i,j})$ is not weakly right noetherian since it has is an infinite antichain $\{y_i S^1 : i \in \mathbb{N}\}$.

Proposition. Let $S = \mathcal{S}(Y, S_\alpha)$ be completely regular. If Y satisfies ACCP, then S satisfies ACCPR.

Remark. There exists $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$, where each S_α is a left zero semigroup, such that S satifies ACCPR but Y does not.

Thanks for listening